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TRACE FORMULAS FOR A CLASS OF TOEPLITZ-LIKE OPERATORS

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ABSTRACT

Let P_T denote projection onto the space of entire functions of exponential type $\leqq T$ which are square summable on the line relative to a measure $d\Delta$ and let G denote multiplication by a suitably restricted complex valued function g. For a reasonably large class of measures $d\Delta$, which includes Lebesgue measure $d\gamma$, it is shown that trace $\{(P_TGP_T)^n - P_TG^nP_T\}$ tends boundedly to a limit as $T \uparrow \infty$ and that the limit is *independent* of the choice of $d\Delta$ within the permitted class. This extends the range of validity of a formula due to Mark Kac who evaluated this limit in the special case $d\Delta = d\gamma$ using a different formalism.

1. Introduction

In this paper we shall study the limit as $T \uparrow \infty$ of the trace of a class of Toeplitz-like operators of the form $(P_TGP_T)^n$ in which G stands for the operator of multiplication, by a suitably restricted complex valued function g, and P_T is the orthogonal projection of $L^2(R^1, d\Delta)$ onto the space $I^T(d\Delta)$ of entire functions of exponential type $\leqq T$ which are square summable relative to the measure $d\Delta(\gamma) = |h(\gamma)|^2 d\gamma$. We shall assume throughout that

$$\int \left| \frac{\log |h(\gamma)|}{\gamma^2+1} \right| d\gamma < \infty$$

and *shall always take h itself to be an outer function.* This is accomplished by defining

$$h(a) = \lim_{b \downarrow 0} h(\omega)$$

in which $\omega = a + ib$ and

$$h(\omega) = \exp\left\{\frac{1}{\pi i}\int \frac{1+\gamma\omega}{\gamma-\omega}\frac{\log|h(\gamma)|}{\gamma^2+1}\, d\gamma\right\}. \tag{1.1}$$

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The indicated limit exists pointwise a.e.; see e.g. page 51 of Dym–McKean [9] for a proof. In additon we shall assume for the moment that

(1.2) *there exists a choice of* $R \geqq 0$ *such that* $h_R(\gamma) = e^{i\gamma R}h(\gamma)$ *agrees a.e. on the real axis with the reciprocal of an entire function of exponential type* $\leqq R$,

although much of the analysis will be carried out under the less restrictive assumption that

(1.3) *there exists a choice of* $R \geqq 0$ *such that* $h_R/h_R^\#$ *agrees a.e. on the real axis with an inner function.*

AMPLIFICATION. If (1.2) is in force and if the exponential type of $1/h_R$ is $\leqq R$, then in fact equality must prevail since $-b^{-1}\log|h_R(ib)| \to R$ as $b \uparrow \infty$. This follows easily from the formulas displayed in the proof of Lemma 2.1.

NOTATION. The limits of integration in the above integrals and all other such unmarked integrals are $\pm\infty$; ω^* stands for the complex conjugate of the complex number ω, whereas G^* will stand for the adjoint of the operator G;

$$h^\#(\omega) = [h(\omega^*)]^*, \quad h_T(\omega) = e^{i\omega T}h(\omega) \quad \text{and} \quad h_T^\#(\omega) = e^{-i\omega T}h^\#(\omega).$$

The space $I^T(d\Delta)$ is a proper closed subspace of $L^2(R^1, d\Delta)$ for every choice of $T \geqq 0$ because of the summability condition imposed on $\log|h|$. Moreover, because of (1.2), the phase $\gamma T + \vartheta(\gamma)$ of

$$h_T(\gamma) = |h(\gamma)|\, e^{i[\gamma T + \varphi(\gamma)]}$$

has non-negative slope $T + \vartheta'(\gamma)$ on R^1 for $T \geqq R$ (see Corollary 2.2). A principal conclusion of this paper can now be expressed conveniently in terms of ϑ' as follows:

If (1.2) *is in effect, and*

$$\int |g(\gamma)|[T + \vartheta'(\gamma)]d\gamma < \infty$$

for every choice of $T \geqq R$, *if* g *is bounded and uniformly continuous and if the inverse Fourier transform*

$$g^\vee(x) = \frac{1}{2\pi}\int g(\gamma)e^{-i\gamma x}d\gamma$$

is subject to the constraint

$$\int |x|\,|g^\vee(x)|^2 dx < \infty,$$

then $(P_TGP_T)^n$ *and* $P_TG^nP_T$ *are of trace class for every* $T \geqq 0$ *and every positive integer* n *and*

$$\text{trace}\{(P_TGP_T)^n - P_TG^nP_T\} = \text{trace}(P_TGP_T)^n - \frac{1}{\pi}\int [g(\gamma)]^n[T+\vartheta'(\gamma)]d\gamma$$
$$(1.4)\qquad = -n\sum_{k=1}^{n-1}\int_0^\infty x\left(\frac{g^k}{k}\right)^\vee(x)\left(\frac{g^{n-k}}{n-k}\right)^\vee(-x)dx + o(1)$$

as $T \uparrow \infty$. *Moreover, for suitably small* ε, *the determinant*

$$\det[I-\varepsilon P_TGP_T] \equiv \exp\left\{-\text{trace}\sum_{n=1}^\infty \frac{[\varepsilon P_TGP_T]^n}{n}\right\}$$
$$(1.5)\qquad = \exp\left\{\frac{1}{\pi}\int \log[1-\varepsilon g(\gamma)][T+\vartheta'(\gamma)]d\gamma\right.$$
$$\left. + \int_0^\infty x(\log[1-\varepsilon g])^\vee(x)(\log[1-\varepsilon g])^\vee(-x)dx + o(1)\right\}$$

as $T \uparrow \infty$.

Formulas (1.4) and (1.5) were first established in a different formalism in the special case $h=1$ by Mark Kac [15]. Kac took g to be real valued and even and assumed that $\int |x||g^\vee(x)|dx < \infty$. These restrictions on g were subsequently relaxed by Baxter [3], Hirschman [11], [12], [13], and Devinatz [6], [7]; see also Akhiezer [1] for a different approach, Kac [16] for comments thereon and Hirschman [14] for a survey and an extensive bibliography. To compare the present results with those of Kac you have only to notice that if $h=1$, then $\vartheta'=0$ and

$$\text{trace}(P_TGP_T)^n$$
$$(1.6)\qquad = \int_{-T}^{T}\cdots\int_{-T}^{T} g^\vee(x_1-x_2)\cdots g^\vee(x_{n-1}-x_n)g^\vee(x_n-x_1)dx_1\cdots dx_n.$$

Identity (1.6) follows from the classical Paley–Wiener theorem which enables you to express P_Tf explicitly in terms of $f^\vee$ in case $h=1$:

$$(P_Tf)(\xi) = \int_{-T}^{T} f^\vee(x)e^{i\xi x}dx = \int_{-\infty}^{\infty}\frac{\sin T(\xi-\eta)}{\pi(\xi-\eta)}f(\eta)d\eta$$

for $f \in L^2(R^1, d\gamma)$. Consequently you see that

$$(P_TGP_Tf)(\xi) = \int_{-\infty}^{\infty} K(\xi,\eta)f(\eta)d\eta$$

in which the kernel

$$K(\xi,\eta)=\int \frac{\sin T(\xi-\gamma)}{\pi(\xi-\gamma)}\, g(\gamma)\, \frac{\sin T(\gamma-\eta)}{\pi(\gamma-\eta)}\, d\gamma,$$

and identity (1.6) is now easily deduced from the fact that

$$\operatorname{trace}(P_TGP_T)^n=\int_{-\infty}^{\infty}\cdots\int K(\xi_1,\xi_2)\cdots K(\xi_{n-1},\xi_n)K(\xi_n,\xi_1)d\xi_1\cdots d\xi_n.$$

The strategy of this paper is to show that

$$\lim_{T\uparrow\infty}\left\{\operatorname{trace}(P_TGP_T)^n-\frac{1}{\pi}\int [g(\gamma)]^n[T+\vartheta'(\gamma)]d\gamma\right\}$$

exists and is independent of the choice of h, within the class of h under consideration. This permits you to evaluate the limit by choosing $h=1$ and invoking (the refined version of) Kac's formula. These results are based in part upon a preliminary study of the orthogonal projection U_T of $L^2(R^1, d\Delta)$ onto†

$$M^T(d\Delta)=L^2(R^1,d\Delta)\ominus((h_T^{\#})^{-1}H^{2+}+(h_T)^{-1}H^{2-})=(h_T^{\#})^{-1}H^{2-}\cap(h_T)^{-1}H^{2+}$$

under the less restrictive assumption (1.3), in place of (1.2). In particular it is shown that, for suitably restricted g,

$$(U_TGU_T)^n-U_TG^nU_T$$

is of trace class for every positive integer $n\geqq 1$ and that

$$\lim_{T\uparrow\infty}\operatorname{trace}\{(U_TGU_T)^n-U_TG^nU_T\}=-n\sum_{k=1}^{n-1}\int_0^{\infty}x\left(\frac{g^k}{k}\right)^{\vee}(x)\left(\frac{g^{n-k}}{n-k}\right)^{\vee}(-x)dx$$

independently of the choice of h. The extra assumption (1.2) then permits you to identify P_T and U_T for $T\geqq 0$, and to make the evaluation

$$\operatorname{trace} U_TG^nU_T=\operatorname{trace} P_TG^nP_T=\frac{1}{\pi}\int [g(\gamma)]^n[T+\vartheta'(\gamma)]d\gamma$$

in terms of the phase ϑ of h for $T\geqq R$.

The cognoscenti will perhaps recognize that, in the presence of (1.2),

$$\frac{1}{\pi}[T+\vartheta'(\gamma)]=J_\gamma^T(\gamma)|h(\gamma)|^2$$

for $T\geqq R$, in which

† $H^{2+}[H^{2-}]$ denotes the Hardy space of class 2 over the upper (lower) half-plane.

$$J_{\beta}^{T}(\gamma)=\frac{e(T,\beta)^{*}e(T,\gamma)-e(T,\beta^{*})e^{\#}(T,\gamma)}{-2\pi i(\gamma-\beta^{*})}$$

is the reproducing kernel for the de Branges space $\boldsymbol{B}(e)$ alias $\boldsymbol{I}^{T}(d\Delta)$ based upon the function

$$e(T,\gamma)=[h_{T}(\gamma)]^{-1}.$$

This suggests that formulas akin to (1.4) and (1.5) should hold for the traces of the operators $(P_{T}GP_{T})^{n}$ in case that P_{T} is the projection onto a suitably indexed family of de Branges subspaces $\boldsymbol{B}(e(T,\gamma))$ of $\boldsymbol{L}^{2}(R^{1},d\Delta)$ for an even wider class of measures $d\Delta$ than those considered above. Indeed the main theorems of Section 5 are stated in the language of $J_{\gamma}^{T}(\gamma)$ assuming only that (1.3) is in effect and that $|h|^{-2}$ is locally summable. This is much less restrictive than (1.2), but still deals with a case in which the fundamental de Branges spaces of interest are the spaces $\boldsymbol{I}^{T}(d\Delta)$. The generalizations of the refined Szegö limit theorem on the circle to non-trigonometric polynomials by Davis and Hirschman [4] and by Askey and Wainger [2] can also be put into the de Branges space formalism by making T run through the positive integers and defining $\boldsymbol{B}(e(T,\gamma))$ as a suitably normed de Branges space of polynomials of degree $< T$.

2. Prerequisites

In this section a number of the implications of the assumptions on h are prepared for future use. The first two chapters of Dym–McKean [9] are suggested for supplementary information on the requisite function theory and the Hardy spaces $\boldsymbol{H}^{2+}[\boldsymbol{H}^{2-}]$ over the upper [lower] half-plane.

The first item of business is to show that (1.2) implies (1.3).

LEMMA 2.1. *If h_{R} agrees a.e. on the line with the reciprocal of an entire function of exponential type $\leqq R$, then $h_{R}/h_{R}^{\#}$ agrees a.e. on the line with an inner function.*

PROOF. The Nevanlinna formula (see e.g. pages 22–25 of Dym–McKean [9]) applied to $(h_{R}^{\#})^{-1}$ yields the bound

$$\begin{aligned}-\log|h_{R}^{\#}(a+ib)| &\leqq Rb-\frac{b}{\pi}\int\frac{\log|h_{R}^{\#}(\gamma)|}{(\gamma-a)^{2}+b^{2}}\,d\gamma\\ &=Rb-\frac{b}{\pi}\int\frac{\log|h(\gamma)|}{(\gamma-a)^{2}+b^{2}}\,d\gamma\end{aligned}$$

for $b>0$. At the same time you have

$$\log|h_R(a+ib)| = -Rb + \frac{b}{\pi}\int \frac{\log|h(\gamma)|}{(\gamma-a)^2+b^2}\,d\gamma$$

for $b>0$ since h is presumed to be an outer function; see (1.1). It follows at once that

$$j = h_R/h_R^{\#}$$

is both analytic in the open upper half-plane and subject to the bound

$$\log|j(a+ib)| \leqq 0$$

for $b \geqq 0$, with equality for $b=0$. Therefore j is an inner function and the proof is complete.

The next item of business is to examine the relationship between the spaces $\boldsymbol{I}^T(d\Delta)$ and $\boldsymbol{M}^T(d\Delta)$.

Lemma 2.2. *$\boldsymbol{I}^T(d\Delta) \subset \boldsymbol{M}^T(d\Delta)$ for every $T \geqq 0$.*

Proof. Let $f \in \boldsymbol{I}^T(d\Delta)$. Then

$$\int \frac{\log^+|f(\gamma)|}{\gamma^2+1}\,d\gamma \leqq \int \frac{\log^+|f(\gamma)h(\gamma)|}{\gamma^2+1}\,d\gamma + \int \frac{\log^+|1/h(\gamma)|}{\gamma^2+1}\,d\gamma$$
$$< \infty$$

and so the Nevanlinna representation formula may be applied to $f_T(\gamma) = e^{i\gamma T}f(\gamma)$ to deduce the bound

$$\log|f_T(a+ib)| \leqq \frac{b}{\pi}\int \frac{\log|f(\gamma)|}{(\gamma-a)^2+b^2}\,d\gamma$$

for $b \geqq 0$. At the same time

$$\log|h(a+ib)| = \frac{b}{\pi}\int \frac{\log|h(\gamma)|}{(\gamma-a)^2+b^2}\,d\gamma$$

for $b>0$, since h is an outer function, and so

$$|(f_Th)(a+ib)|^2 \leqq \exp\left\{\frac{b}{\pi}\int \frac{\log|(fh)(\gamma)|^2}{(\gamma-a)^2+b^2}\,d\gamma\right\}$$

for $b>0$, whence $fh_T = f_Th$ is seen to belong to $\boldsymbol{H}^{2+}$:

$$\int |(fh_T)(a+ib)|^2 da \leqq \int |(fh)(a)|^2 d\gamma$$
$$= \|f\|_\Delta^2$$

independently of $b>0$. It follows readily that f is orthogonal to $(h_T)^{-1}\boldsymbol{H}^{2-}$ in

$L^2(R^1, d\Delta)$ as is $f^\#$, by the same argument. Therefore f is orthogonal to $(h_T^\#)^{-1}H_2^+$ also and so must belong to $M^T(d\Delta)$. The proof is complete.

AMPLIFICATION. It has already been noted that *$I^T(d\Delta)$ is a proper closed subspace of $L^2(R^1, d\Delta)$ for every $T \geqq 0$.* A proof may be patterned on the argument given on p. 316 of Dym–McKean [8]; see also page 151 of Pitt [17]. The estimates derived in the verification of Lemma 2.2 come into play in the former.

THEOREM 2.1. *If $|h|^{-2}$ is locally summable, then $M^T(d\Delta) = I^T(d\Delta)$ for every $T \geqq 0$.*

PROOF. Fix $T \geqq 0$ and choose $f \in M^T(d\Delta)$. In view of Lemma 2.2 you have only to show that f can be identified with the restriction to R^1 of an entire function of exponential type $\leqq T$. Since $fh_T \in H^{2+}$ and $fh_T^\# \in H^{2-}$ you may presume from the outset that f is defined and analytic in both the open upper half-plane and the open lower half-plane and that

$$f(a) = \lim_{b \downarrow 0} f(a+ib) = \lim_{b \downarrow 0} f(a-ib) \qquad \text{a.e.}$$

Moreover, f is locally summable:

$$\left(\int_{-c}^{c} |f(a)|\,da\right)^2 \leqq \int_{-c}^{c} |(fh)(a)|^2 da \int_{-c}^{c} |h(a)|^{-2} da < \infty$$

for $0 < c < \infty$, and

$$\begin{aligned} 2\int \frac{\log^+|f(a)|}{a^2+1}\,da &\leqq \int \frac{\log^+|(fh)(a)|^2}{a^2+1}\,da + \int \frac{\log^+|h(a)|^{-2}}{a^2+1}\,da \\ &\leqq \int \frac{|(fh)(a)|^2}{a^2+1}\,da + 2\int \left|\frac{\log|h(a)|}{a^2+1}\right| da \\ &< \infty. \end{aligned}$$

The proof that f may be identified with an entire function of exponential type $\leqq T$ is now completed by an argument due to Levinson–McKean; see pages 115–116 of Dym–McKean [9] for the details. The type estimate involves a minor modification of the arguments given there and so the main ideas will be sketched. The first step is to extract the bound

$$\begin{aligned} \log|f(Re^{i\theta})| &\leqq TR\,|\sin\theta| + \frac{R\,|\sin\theta|}{\pi}\int \frac{\log^+|f(\gamma)|}{|\gamma - Re^{i\theta}|^2}\,d\gamma \\ &\leqq TR\,|\sin\theta| + \frac{R\,|\sin\theta|}{\pi[1-\cos\theta]}\int \frac{\log^+|f(\gamma)|}{\gamma^2+R^2}\,d\gamma, \end{aligned}$$

for $0<|\theta|<\pi$, from the fact that $fh_T\in \boldsymbol{H}^{2+}$, $fh_T^{\#}\in \boldsymbol{H}^{2-}$ and h is an outer function. This implies that f is of exponential type $\leqq T$ in the two sectors $\pi/9\leqq|\theta|\leqq 8\pi/9$ and that $f(\gamma)e^{-\gamma T}$ $[f(\gamma)e^{\gamma T}]$ is bounded on the rays $\gamma = Re^{i\theta}: \theta = \pm\pi/9$ $[\theta = \pm 8\pi/9]$. But now f is of order $\leqq 4$, as is shown in the reference cited above, and $2\pi/9<\pi/4$, and so the Phragmén–Lindelöf principle implies that $f(\gamma)e^{-\gamma T}$ $[f(\gamma)e^{\gamma T}]$ is bounded in the sector $|\theta|\leqq\pi/9$ $[|\pi-\theta|\leqq\pi/9]$. Thus f is seen to be of exponential type $\leqq T$ in the whole complex plane, and the proof is complete.

Notation. $\langle\,,\rangle_\Delta$ $[\|\ \|_\Delta]$ stands for the inner product [norm] in $L^2(R^1, d\Delta)$.

Theorem 2.2. *If $I^T(d\Delta)\neq 0$, then it is a reproducing kernel Hilbert space with reproducing kernel*

$$J_\omega^T(\gamma)=\frac{e(T,\omega)^*e(T,\gamma)-e^{\#}(T,\omega)^*e^{\#}(T,\gamma)}{-2\pi i(\gamma-\omega^*)} \tag{2.1}$$

based upon the (de Branges) function

$$e=e(T,\cdot\,)=\frac{P_T^\circ(1/h_T)}{\|P_T^\circ(1/h_T)\|_{\Delta^\circ}}. \tag{2.2}$$

in which $d\Delta^\circ(\gamma)=[\pi(\gamma^2+1)]^{-1}d\Delta(\gamma)$ and P_T° is the orthogonal projection of $L^2(R^1, d\Delta^\circ)$ onto $I^T(d\Delta^\circ)$.

Proof. There are two things to show: that $J_\omega^T\in I^T(d\Delta)$ for every complex ω and that

$$f(\omega)=\langle f, J_\omega^T\rangle_\Delta \tag{2.3}$$

for every complex ω and every $f\in I^T(d\Delta)$. The first is self-evident once you know that e is well defined. But if $I^T(d\Delta)\neq 0$, then there exists a function $f\in I^T(d\Delta^\circ)$ with $f(i)\neq 0$ and so the Cauchy formula for $\boldsymbol{H}^{2+}$ functions, applied to $(1-i\gamma)^{-1}fh_T$, implies that

$$\langle f, 1/h_T\rangle_{\Delta^\circ}=\frac{1}{\pi}\int\frac{f(\gamma)h_T(\gamma)}{\gamma^2+1}\,d\gamma=f(i)h_T(i)\neq 0.$$

Hence $\|P_T^\circ(1/h_T)\|_{\Delta^\circ}\neq 0$ and e is well defined. Since e itself belongs to $I^T(d\Delta^\circ)$ you see that

$$1=\langle e, e\rangle_{\Delta^\circ}=\frac{e(T,i)h_T(i)}{\|P_T^\circ(1/h_T)\|_{\Delta^\circ}}.$$

This proves that $e(T,i)>0$, since $h_T(i)>0$, and yields the identities

$$\langle f, e \rangle_{\Delta^\circ} = \frac{f(i)}{e(T,i)}$$

and

$$\langle f, e^{\#} \rangle_{\Delta^\circ} = [\langle f^{\#}, e \rangle_{\Delta^\circ}]^* = \left[\frac{f^{\#}(i)}{e(T,i)}\right]^* = \frac{f(-i)}{e(T,i)}$$

for $f \in \boldsymbol{I}^T(d\Delta^\circ)$, and the subsequent evaluation

$$\begin{aligned} \|J_i^T\|_\Delta^2 &= \int \left| \frac{e(T,i)^* e(T,\gamma) - e^{\#}(T,i)^* e^{\#}(T,\gamma)}{\pi(\gamma - i)} \right|^2 d\Delta(\gamma) \\ &= \frac{|e(T,i)|^2 - |e(T,-i)|^2}{4\pi} \\ &= J_i^T(i) > 0. \end{aligned}$$

The last inequality follows from the fact that

$$\left| \frac{e(T,-i)}{e(T,i)} \right| = |\langle e^{\#}, e \rangle_{\Delta^\circ}| \leqq \|e^{\#}\|_{\Delta^\circ} \|e\|_{\Delta^\circ} = 1$$

with equality if and only if

$$e^{\#} = ce$$

with a constant c of modulus 1. But that in turn would imply that

$$f(i) = cf(-i)$$

for every $f \in \boldsymbol{I}^T(d\Delta^\circ)$ which is clearly not the case if $\boldsymbol{I}^T(d\Delta) \neq 0$.

Now if f and g belong to $\boldsymbol{I}^T(d\Delta)$ and ω is fixed, then

$$\left[\frac{f(\gamma)g(\omega) - f(\omega)g(\gamma)}{\gamma - \omega}\right](\gamma^2 + 1)$$

belongs to $\boldsymbol{I}^T(d\Delta^\circ)$ and is orthogonal to both e and $e^{\#}$ in $\boldsymbol{L}^2(R^1, d\Delta^\circ)$. Therefore

$$\begin{aligned} &e(T,\omega) \int \frac{f(\gamma)g(\omega) - f(\omega)g(\gamma)}{2\pi i(\gamma - \omega)} e(T,\gamma)^* d\Delta(\gamma) \\ &\quad - e^{\#}(T,\omega) \int \frac{f(\gamma)g(\omega) - f(\omega)g(\gamma)}{2\pi i(\gamma - \omega)} e^{\#}(T,\gamma)^* d\Delta(\gamma) = 0, \end{aligned}$$

or, what amounts to the same,

$$g(\omega)\langle f, J_\omega^T \rangle_\Delta = f(\omega)\langle g, J_\omega^T \rangle_\Delta.$$

The choice $g = J_\omega^T$ yields the identity

$$J_\omega^T(\omega)\langle f, J_\omega^T\rangle_\Delta = f(\omega)\|J_\omega^T\|_\Delta^2.$$

Now suppose that $f(i) \neq 0$. Then since $J_i^T(i) > 0$, you may choose a small disc about the point i such that

$$\frac{\langle f, J_\omega^T\rangle_\Delta}{f(\omega)} = \frac{\|J_\omega^T\|_\Delta^2}{J_\omega^T(\omega)}$$

is both analytic and real valued (in fact positive) for all points ω in that disc. Therefore, by the open mapping theorem, there is a constant c such that

$$f(\omega) = c\langle f, J_\omega^T\rangle_\Delta$$

for all points ω in that disc, and hence for all points ω in the complex plane since both sides of the equality are entire functions. The choice $f = J_i^T$ and $\omega = i$ implies that $c = 1$, and so (2.3) is established for those $f \in \boldsymbol{I}^T(d\Delta)$ with $f(i) \neq 0$. But now if $f(i) = 0$ you may take $g \in \boldsymbol{I}^T(d\Delta)$ with $g(i) \neq 0$ and apply (2.3) to $f + g$ and to g separately. Hence, by linearity (2.3) is valid for all $f \in \boldsymbol{I}^T(d\Delta)$, and the proof is complete.

Amplification. $\boldsymbol{I}^T(d\Delta)$ may be identified as the de Branges space $\boldsymbol{B}(e)$ based upon the de Branges function e. See Dym–McKean [9] for an introduction to such spaces and de Branges [5] for a more comprehensive treatment. The identification (2.2) of e in terms of a projection is adapted from page 315 of Dym–McKean [8].

Corollary 2.1. *If ϕ_k, $k = 1, 2, \cdots$, is an orthonormal basis for $\boldsymbol{I}^T(d\Delta)$, then*

$$\sum_{k=1}^{\infty} |\phi_k(\omega)|^2 = J_\omega^T(\omega) \tag{2.4}$$

for every complex number ω.

Proof. A double application of (2.3) with $f = J_\omega^T$ coupled with the Plancherel formula yields the result:

$$\begin{aligned} J_\omega^T(\omega) &= \langle J_\omega^T, J_\omega^T\rangle_\Delta \\ &= \sum_{k=1}^{\infty} |\langle \phi_k, J_\omega^T\rangle_\Delta|^2 \\ &= \sum_{k=1}^{\infty} |\phi_k(\omega)|^2. \end{aligned}$$

Corollary 2.2. *If h_R agrees a.e. on the line with the reciprocal of an entire function of exponential type $\leqq R$, then $\boldsymbol{M}^T(d\Delta) = \boldsymbol{I}^T(d\Delta)$ for $T \geqq 0$,* (2.5) $$e(T,\gamma)=[h_T(\gamma)]^{-1},$$

for $T \geqq R$, *and*

(2.6) $$J^T_\gamma(\gamma)=\frac{|h(\gamma)|^{-2}}{\pi}[T+\vartheta'(\gamma)],$$

for $T \geqq R$ *and* $\gamma \in R^1$, *in which* ϑ *denotes the phase of* h.

PROOF. The first two assertions are an immediate consequence of Theorems 2.1 and 2.2. Formula (2.6) then follows upon substituting (2.5) into (2.1) and evaluating the resulting expression with $\omega = \gamma$ real.

3. Preliminary estimates

In this section a number of preliminary estimates related to the growth of the trace of $(U_TGU_T)^n$ as $T \uparrow \infty$ will be derived. Gohberg–Krein [10] is recommended as a general source of information on trace class (alias nuclear) and Hilbert–Schmidt operators.

NOTATION. $\|A\|$, $|A|_1$ and $|A|_2$ stand for the usual operator norm, the trace class norm (i.e., the sum of the s-numbers) and the Hilbert–Schmidt norm of the operator A, respectively.

LEMMA 3.1. *If* $f \in \boldsymbol{M}^T(d\Delta)$, *then*

$$(fh)^\vee(x)=0 \quad \textit{for} \quad x<-T$$

and

$$(fh^{\#})^\vee(x)=0 \quad \textit{for} \quad x>T.$$

PROOF. If $f \in \boldsymbol{M}^T(d\Delta)$, then $fh_T \in \boldsymbol{H}^{2+}$ and $fh^{\#}_T \in \boldsymbol{H}^{2-}$. Therefore

$$(fh_T)^\vee(x)=0 \quad \text{for} \quad x<0$$

and

$$(fh^{\#}_T)^\vee(x)=0 \quad \text{for} \quad x>0.$$

The rest is plain.

Let $\mathfrak{p}[\mathfrak{q}]$ denote the orthogonal projection of $\boldsymbol{L}^2(R^1,d\gamma)$ onto $\boldsymbol{H}^{2+}$ $[\boldsymbol{H}^{2-}]$:

$$\mathfrak{p}: f \in \boldsymbol{L}^2(R^1,d\gamma) \to \int_0^\infty f^\vee(x)e^{i\gamma x}dx,$$

$$\mathfrak{q}: f \in \boldsymbol{L}^2(R^1,d\gamma) \to \int_{-\infty}^0 f^\vee(x)e^{i\gamma x}dx$$

and let $V = V_T[W = W_T]$ denote the orthogonal projection of $\boldsymbol{L}^2(R^1, d\Delta)$ onto $(h_T^{\#})^{-1}\boldsymbol{H}^{2+}$ $[h_T^{-1}\boldsymbol{H}^{2-}]$. Then it is readily checked that

$$V_T f = (h_T^{\#})^{-1}\mathfrak{p}h_T^{\#}f$$

and

$$W_T f = (h_T)^{-1}\mathfrak{q}h_T f,$$

for $f \in \boldsymbol{L}^2(R^1, d\Delta)$.

WARNING. The dependence of the projections P_T, U_T, V_T and W_T upon T is often suppressed in order to simplify the typography.

LEMMA 3.2. *If* (1.3) *is in effect and* $T \geqq R$, *then* $V_TW_T = W_TV_T = 0$ *and*

$$U_T = I - (V_T + W_T).$$

PROOF. By assumption $h_T/h_T^{\#}$ is an inner function for $T = R$ and so too for $T \geqq R$. Therefore, since $\boldsymbol{H}^{2+}$ is both closed under multiplication by inner functions and orthogonal to $\boldsymbol{H}^{2-}$, you see that

$$\begin{aligned} W_TV_Tf &= (h_T)^{-1}\mathfrak{q}(h_T/h_T^{\#})\mathfrak{p}h_T^{\#}f \\ &= 0 \end{aligned}$$

for $T \geqq R$ and $f \in \boldsymbol{L}^2(R^1, d\Delta)$. This proves that

$$WV = W_TV_T = 0$$

and hence that

$$VW = V^*W^* = (WV)^* = 0$$

also. The rest is plain.

LEMMA 3.3. *If* $f \in \boldsymbol{L}^2(R^1, d\Delta)$, *then*

$$\| V_T f \|_\Delta^2 = 2\pi \int_T^\infty |(fh^{\#})^\vee(x)|^2 dx = o(1), \qquad as \quad T \uparrow \infty,$$

and

$$\| W_T f \|_\Delta^2 = 2\pi \int_{-\infty}^{-T} |(fh)^\vee(x)|^2 dx = o(1), \qquad as \quad T \uparrow \infty.$$

If also (1.3) *is in effect and* $T \geqq R$, *then*

$$\| U_T f - f \|_\Delta^2 = \| V_T f + W_T f \|_\Delta^2 = \| V_T f \|_\Delta^2 + \| W_T f \|_\Delta^2 = o(1).$$

PROOF. By the classical Plancherel formula

$$\| W_T f \|_\Delta^2 = \int_{-\infty}^{\infty} |(qh_T f)(\gamma)|^2 d\gamma = \int_{-\infty}^{\infty} \left| \int_{-\infty}^{0} (fh_T)^\vee(x) e^{i\gamma x} dx \right|^2 d\gamma$$

$$= 2\pi \int_{-\infty}^{0} |(fh_T)^\vee(x)|^2 dx$$

$$= 2\pi \int_{-\infty}^{-T} |(fh)^\vee(x)|^2 dx$$

which clearly tends to zero as $T \uparrow \infty$ since $(fh)^\vee \in L^2(R^1, dx)$. The evaluation of $\| Vf \|_\Delta$ is similar and the final statement drops out from the fact that V and W are mutually orthogonal projections for $T \geqq R$.

LEMMA 3.4. *If g is a bounded function of class $L^2(R^1, d\gamma)$, then the Hilbert–Schmidt norms of $V_T G U_T$ and $W_T G U_T$ are subject to the bounds*

$$| V_T G U_T |_2^2 \leqq \tau = \int_0^{\infty} x |g^\vee(x)|^2 dx,$$

$$| W_T G U_T |_2^2 \leqq \tau' = \int_{-\infty}^{0} |x| |g^\vee(x)|^2 dx.$$

If also (1.3) *is in effect and $T \geqq R$, then*

$$|(V_T + W_T) G U_T|_2^2 \leqq \tau + \tau'.$$

PROOF. Let ϕ_k, $k = 1, 2, \cdots$, be any orthonormal basis for $M^T(d\Delta)$. Then

$$\| VG\phi_k \|_\Delta^2 = 2\pi \int_T^{\infty} |(g\phi_k h^\#)^\vee(x)|^2 dx$$

by Lemma 3.3 and so as $(\sqrt{2\pi}\phi_k h^\#)^\vee$ is an orthonormal system of functions in $L^2[(-\infty, T], dx)$ [see Lemma 3.1] and

$$\int_T^{\infty} |(g\phi_k h^\#)^\vee(x)|^2 dx = \int_T^{\infty} dx \left| \int_{-\infty}^{T} g^\vee(x-y)(\phi_k h^\#)^\vee(y) dy \right|^2$$

you see that

$$| VGU |_2^2 = \sum_{k=1}^{\infty} \| Vg\phi_k \|_\Delta^2 \leqq \int_T^{\infty} dx \int_{-\infty}^{T} |g^\vee(x-y)|^2 dy$$

$$= \int_0^{\infty} dx \int_0^{\infty} |g^\vee(x+y)|^2 dy = \int_0^{\infty} x |g^\vee(x)|^2 dx,$$

as advertised. The second inequality is proved in much the same way, and the last is a simple consequence of the first two and the mutual orthogonality of V and W for $T \geqq R$.

LEMMA 3.5. *If g is a bounded function of class $L^2(R^1, d\gamma)$, and if* (1.3) *is in effect and $S = T - R \geqq 0$, then*

$$\| V_T G W_T \|^2 \leqq \int_{2S}^{\infty} [y - 2S] | g^{\vee}(y)|^2 dy$$

and

$$\| W_T G V_T \|^2 \leqq \int_{-\infty}^{-2S} [|y| - 2S] | g^{\vee}(y)|^2 dy.$$

PROOF. If $f \in L^2(R^1, d\Delta)$, and $S = T - R \geqq 0$, then

$$h_T^{\#} W f = (h_T^{\#}/h_T) q h_T f$$

belongs to

$$e^{-i\gamma 2S}(h_R^{\#}/h_R) H^{2-} \subset e^{-i\gamma 2S} H^{2-}$$

since $h_R/h_R^{\#}$ is an inner function. Therefore

$$(h_T^{\#} W f)^{\vee}(y) = 0 \quad \text{for} \quad y > -2S$$

and the desired bound on $V_T G W_T$ is easily achieved, much as in the proof of Lemma 3.4:

$$\begin{aligned}
\| VGWf \|_{\Delta}^2 &= 2\pi \int_0^{\infty} |(g h_T^{\#} W f)^{\vee}(x)|^2 dx \\
&= 2\pi \int_0^{\infty} dx \left| \int_{-\infty}^{-2S} g^{\vee}(x - y)(h_T^{\#} W f)^{\vee}(y) dy \right|^2 \\
&\leqq 2\pi \int_0^{\infty} dx \int_{-\infty}^{-2S} | g^{\vee}(x - y)|^2 dy \, \|(h_T^{\#} W f)^{\vee}\|_2^2 \\
&= \int_{2S}^{\infty} [y - 2S] | g^{\vee}(y)|^2 dy \, \| W f \|_{\Delta}^2 \\
&\leqq \int_{2S}^{\infty} [y - 2S] | g^{\vee}(y)|^2 dy \, \| f \|_{\Delta}^2,
\end{aligned}$$

since W is a projection. This completes the proof of the first inequality. The second is proved in much the same way.

LEMMA 3.6. *If g is a bounded function of class $L^2(R^1, d\gamma)$, and if* (1.3) *is in effect and $T \geqq R$, then*

$$|(V_T + W_T)G^m U_T|_2 \leqq m \, \|G\|^{m-1}[\tau + \tau']^{1/2}.$$

PROOF. Let

$$\alpha_m = |(V + W)G^m U|_2$$

for $m = 1, 2, \cdots$. Then, for $m > 1$,

$$\begin{aligned} \alpha_m &= |(V + W)G(U + V + W)G^{m-1}U|_2 \\ &\leqq |(V + W)GU|_2 \|G^{m-1}U\| + \|(V + W)G\| \, |(V + W)G^{m-1}U|_2 \\ &\leqq \|G\|^{m-1}\alpha_1 + \|G\| \, \alpha_{m-1}, \end{aligned}$$

since $V + W$ and U are projections, and so, by a simple inductive argument,

$$\alpha_m \leqq m \, \|G\|^{m-1}\alpha_1.$$

But this completes the proof since

$$\alpha_1 \leqq [\tau + \tau']^{1/2},$$

by Lemma 3.4.

LEMMA 3.7. *If g is a bounded function of class $L^2(R^1, d\gamma)$, then*

$$\int |y| \, |(g^m)^\vee(y)|^2 dy \leqq m^2 \|G\|^{2(m-1)}[\tau + \tau'].$$

PROOF. Choose $h = 1$ so that $d\Delta = d\gamma$ and $M^T(d\Delta) = I^T(d\Delta)$. Then, by a routine calculation, much as in the proof of Lemma 3.4, you find that

$$\begin{aligned} \int_{-\infty}^{-2T} [|y| - 2T] |(g^m)^\vee(y)|^2 dy &+ \int_{2T}^{\infty} [y - 2T] |(g^m)^\vee(y)|^2 dy \\ &= |W_T G^m U_T|_2^2 + |V_T G^m U_T|_2^2 \\ &= |(V_T + W_T)G^m U_T|_2^2 \\ &\leqq m^2 \|G\|^{2(m-1)}[\tau + \tau'] \end{aligned}$$

for every $T > 0$. The bound in the last line comes from Lemma 3.6. It is independent of T and so you have only to let $T \downarrow 0$ in the integrals to complete the proof.

Now fix an orthonormal basis ψ_k, $k = 1, 2, \cdots$, of $M^T(d\Delta)$ for $T - R = S \geqq 0$ and introduce the functions

$$u_k(x) = \begin{cases} \sqrt{2\pi}(h_R\psi_k)^\vee(x) & \text{for} \quad |x| \leqq S \\ 0 & \text{for} \quad |x| > S \end{cases}$$

and

$$v_k(x) = \begin{cases} \sqrt{2\pi}(h_R\psi_k)^\vee(x) & \text{for} \quad x > S \\ 0 & \text{for} \quad x \leqq S. \end{cases}$$

Because of Lemma 3.1

$$u_k + v_k = \sqrt{2\pi}(h_R\psi_k)^\vee(x).$$

THEOREM 3.1. *If $f \in L^2(R^1, dx)$, and if (1.3) is in effect and $S = T - R \geqq 0$, then*

$$\sum_{k=1}^{\infty} |\langle f, u_k \rangle|^2 = \int_{-S}^{S} |f(x)|^2 dx$$

and

$$\sum_{k=1}^{\infty} |\langle f, v_k \rangle|^2 \leqq \int_{S}^{\infty} |f(x)|^2 dx.$$

NOTATION. $\langle\,,\rangle[\|\ \|]$ denotes the standard inner product (norm) in L^2; $\hat{f}$ denotes the usual Fourier transform for f in $L^2(R^1, dx)$: $\hat{f}(\gamma) = \int f(x)e^{i\gamma x}dx$.

PROOF. Let

$$f_0(x) = \begin{cases} f(x) & \text{for} \quad |x| \leqq S \\ 0 & \text{for} \quad |x| > S. \end{cases}$$

Then

$$\begin{aligned} 2\pi \sum |\langle f, u_k \rangle|^2 &= 2\pi \sum |\langle f_0, (\sqrt{2\pi}h_R\psi_k)^\vee \rangle|^2 \\ &= \sum |\langle \hat{f}_0, h_R\psi_K \rangle|^2 \\ &= \sum |\langle \hat{f}_0/h_R, \psi_k \rangle_\Delta|^2 \\ &= \| U_T(\hat{f}_0/h_R)\|_\Delta^2 \\ &= \|(\hat{f}_0/h_R)\|_\Delta^2 - \| W_T(\hat{f}_0/h_R)\|_\Delta^2 - \| V_T(\hat{f}_0/h_R)\|_\Delta^2. \end{aligned}$$

But now

$$\|(\hat{f}_0/h_R)\|_\Delta^2 = \|\hat{f}_0\|^2 = 2\pi\int_{-S}^{S} |f(x)|^2 dx,$$

whereas, by Lemma 3.3,

$$\begin{aligned}\| W_T(\hat{f}_0/h_R)\|_\Delta^2 &= 2\pi\int_{-\infty}^{-T} |(\hat{f}_0 h/h_R)^\vee(x)|^2 dx \\ &= 2\pi\int_{-\infty}^{-T} |f_0(x+R)|^2 dx \\ &= 2\pi\int_{-\infty}^{-S} |f_0(x)|^2 dx = 0,\end{aligned}$$

and

$$\begin{aligned}\| V_T(\hat{f}_0/h_R)\|_\Delta^2 &= 2\pi\int_{T}^{\infty} |(\hat{f}_0 h^\#/h_R)^\vee(x)|^2 dx \\ &= 2\pi\int_{T}^{\infty} |(\hat{f}_0 h_R^\#/h_R)^\vee(x-R)|^2 dx \\ &= 2\pi\int_{S}^{\infty} |(\hat{f}_0 h_R^\#/h_R)^\vee(x)|^2 dx = 0;\end{aligned}$$

the final evaluation depends upon the fact that $h_R/h_R^\#$ is an inner function. This completes the proof of the first assertion. The second is easier:

$$\begin{aligned}\sum_{k=1}^{\infty} |\langle f, v_k\rangle|^2 &= \sum_{k=1}^{\infty} \left| \int_S^\infty f(x) v_k(x)^* dx \right|^2 \\ &= \sum_{k=1}^{\infty} \left| \sqrt{2\pi}\int_S^\infty f(x)[(h_R\psi_k)^\vee(x)]^* dx \right|^2 \\ &\leqq \int_S^\infty |f(x)|^2 dx,\end{aligned}$$

since the functions $\sqrt{2\pi}(h_R\psi_k)^\vee$ are orthonormal in $L^2(R^1, dx)$. The theorem is proved.

COROLLARY 3.1. *If $f \in L^2(R^1, dx)$ is bounded, and if* (1.3) *is in effect and $S = T - R \geqq 0$, then*

$$\sum_{k=1}^{\infty} \| \mathfrak{q} f e^{i\gamma S}\hat{u}_k \|^2 \leqq 2\pi\int_{-\infty}^{0} |x|\,|f^\vee(x)|^2 dx,$$

and

$$\sum_{k=1}^{\infty} \| \mathfrak{q} f e^{i\gamma S} \hat{v}_k \|^2 \leqq 2\pi \int_{-\infty}^{-2S} |x| |f^{\vee}(x)|^2 dx.$$

PROOF. The evaluations

$$\| \mathfrak{q} f e^{i\gamma S} \hat{u}_k \|^2 = 2\pi \int_{-\infty}^{-S} dx \left| \int_{-S}^{S} f^{\vee}(x-y) u_k(y) dy \right|^2$$

and

$$\| \mathfrak{q} f e^{i\gamma S} \hat{v}_k \|^2 = 2\pi \int_{-\infty}^{-S} dx \left| \int_{S}^{\infty} f^{\vee}(x-y) v_k(y) dy \right|^2$$

are made just as in the proof of Lemma 3.3. But now, by Theorem 3.1,

$$\begin{aligned} \sum_{k=1}^{\infty} \| \mathfrak{q} f e^{i\gamma S} \hat{u}_k \|^2 &= 2\pi \int_{-\infty}^{-S} dx \int_{-S}^{S} |f^{\vee}(x-y)|^2 dy \\ &= 2\pi \int_{-\infty}^{0} dx \int_{-2S}^{0} |f^{\vee}(x+y)|^2 dy \\ &\leqq 2\pi \int_{-\infty}^{0} |x| |f^{\vee}(x)|^2 dx \end{aligned}$$

and

$$\begin{aligned} \sum_{k=1}^{\infty} \| \mathfrak{q} f e^{i\gamma S} \hat{v}_k \|^2 &\leqq 2\pi \int_{-\infty}^{-S} dx \int_{S}^{\infty} |f^{\vee}(x-y)|^2 dy \\ &= 2\pi \int_{-\infty}^{0} dx \int_{-\infty}^{-2S} |f^{\vee}(x+y)|^2 dy \\ &\leqq 2\pi \int_{-\infty}^{-2S} |y| |f^{\vee}(y)|^2 dy, \end{aligned}$$

as advertised.

4. Principal conclusions for UGU

In this section it will be shown that *if* (1.3) *is in effect and* $T - R = S \geqq 0$, *then*

$$(U_T G U_T)^n - U_T G^n U_T$$

is of trace class for every integer $n \geqq 1$ *and, as* $T \uparrow \infty$, *the trace of this operator tends to a limit which is independent of h providing that g is a bounded function of class* $L^2(R^1, d\gamma)$ *and* $\tau + \tau' < \infty$. The stated assumptions on h and g will be in force throughout this section.

The first step is to make use of the identity $U_T = I - (V_T + W_T)$, which is valid for $T \geqq R$, in order to express $(U_T G U_T)^{n+1}$ as the sum of 2^n terms of the form

$$U_T G L_1 G L_2 \cdots G L_n G U_T$$

in which L_i, $i = 1, \cdots, n$, is equal to either the identity I or to the projection $-(V_T + W_T)$. Each of these is of trace class and the trace of all but one of them stays bounded as $T \uparrow \infty$:

THEOREM 4.1. *If L_i is equal to either I or to $-(V_T + W_T)$, then*

$$U_T G L_1 G \cdots L_n G U_T$$

is of trace class and

$$| U_T G L_1 G \cdots L_n G U_T |_1 \leqq n^2 \| G \|^{n-1} [\tau + \tau']$$

for every $T \geqq R$ and every choice of L_i providing that at least one of the L_i is set equal to $-(V_T + W_T)$.

PROOF. Suppose for the sake of definiteness that exactly two of the L_i are equal to $-(V + W)$. Then

$$UGL_1 G \cdots L_n GU = UG^r (V + W) G^s (V + W) G^t U$$

in which r, s, and t are positive integers which sum to $n + 1$. This exhibits the indicated operator as the product of two Hilbert–Schmidt operators (see Lemma 3.6) and the bounded operator G^s. It is therefore of trace class. Moreover,

$$\begin{aligned} | UGL_1 G \cdots L_n GU |_1 &\leqq | UG^r (V + W) |_2 \| G \|^s | (V + W) G^t U |_2 \\ &\leqq r \| G \|^{r-1} \| G \|^s t \| G \|^{t-1} [\tau + \tau'] \\ &\leqq n^2 \| G \|^{n-1} [\tau + \tau'], \end{aligned}$$

by Lemma 3.6. It remains only to check that the same bound prevails for every permissible choice of the L_i, i.e., whenever one or more of the L_i is chosen equal to $-(V + W)$. But that is easily done with above calculations as a guide.

THEOREM 4.2.

$$\begin{aligned} &| \operatorname{trace}\{(U_T G U_T)^{n+1} - U_T G^{n+1} U_T\} | \\ &\leqq | (U_T G U_T)^{n+1} - U_T G^{n+1} U_T |_1 \\ &\leqq n^2 \| G \|^{n-1} [\tau + \tau'] \end{aligned}$$

for $n = 1, 2, \cdots$, and $T \geqq R$.

PROOF. You have only to decompose $(UGU)^{n+1} - UG^{n+1}U$ into a sum of n distinct pieces of the form

$$-UG^k(V+W)(GU)^{n+1-k}$$

for $k = 1, \cdots, n$, and to extract the asserted inequality from the bound

$$\begin{aligned} |UG^k(V+W)(GU)^{n+1-k}|_1 &\leqq |UG^k(V+W)|_2 |(V+W)GU|_2 \|G\|^{n-k} \\ &\leqq k\|G\|^{n-1}|UG(V+W)|_2|(V+W)GU|_2, \end{aligned}$$

much as in the proof of Theorem 4.1.

THEOREM 4.3. *If $\alpha_0, \alpha_1, \cdots, \alpha_m$ are positive integers which sum to $n+1$:*

$$\alpha_0 + \alpha_1 + \cdots + \alpha_m = n+1,$$

then

$$U_T G^{\alpha_0} W_T G^{\alpha_1} W_T G^{\alpha_2} \cdots W_T G^{\alpha_m} U_T$$

is of trace class for $T \geqq R$ and the limit points of

$$\operatorname{trace}\{U_T G^{\alpha_0} W_T G^{\alpha_1} \cdots W_T G^{\alpha_m} U_T\},$$

as $T \uparrow \infty$, are independent of h.

PROOF. The factorization

$$UG^{\alpha_0}WG^{\alpha_1}\cdots WG^{\alpha_m}U = (UG^{\alpha_0}W)(G^{\alpha_1}\cdots WG^{\alpha_{m-1}})(WG^{\alpha_m}U)$$

exhibits the indicated operators as the product of two Hilbert–Schmidt operators (see Lemma 3.6) and a bounded operator. It is therefore of trace class. Now let ψ_k, $k = 1, 2, \cdots$, be the orthonormal basis for $M^T(d\Delta)$ which was fixed for the proof of Theorem 3.1. Then

$$\begin{aligned} &2\pi\langle WG^{\alpha_1}WG^{\alpha_2}\cdots WG^{\alpha_m}\psi_k, G^{*\alpha_0}\psi_k\rangle_\Delta \\ &= 2\pi\langle \mathfrak{q}G^{\alpha_1}\mathfrak{q}G^{\alpha_2}\cdots \mathfrak{q}G^{\alpha_m}e^{i\gamma T}h\psi_k, G^{*\alpha_0}e^{i\gamma T}h\psi_k\rangle \\ &= \langle \mathfrak{q}G^{\alpha_1}\cdots \mathfrak{q}G^{\alpha_m}e^{i\gamma S}(\hat{u}_k + \hat{v}_k), G^{*\alpha_0}e^{i\gamma S}(\hat{u}_k + \hat{v}_k)\rangle, \end{aligned}$$

and a tedious but elementary calculation based upon the isometry exhibited in Theorem 3.1 shows that

$$\begin{aligned} &\sum_{k=1}^{\infty} \langle \mathfrak{q}G^{\alpha_1}\cdots \mathfrak{q}G^{\alpha_m}e^{i\gamma S}\hat{u}_k, G^{*\alpha_0}e^{i\gamma S}\hat{u}_k\rangle \\ &\qquad = 2\pi\sum_{k=1}^{\infty} \langle \mathfrak{q}G^{\alpha_1}\cdots \mathfrak{q}G^{\alpha_m}e^{i\gamma S}\theta_k, G^{*\alpha_0}e^{i\gamma S}\theta_k\rangle \end{aligned}$$

for any orthonormal basis θ_k, $k = 1,2,\cdots$, of $I^S(d\gamma) = M^S(d\gamma)$. Therefore it remains but to show that the contribution of the terms involving $\hat{v}_k$ becomes negligible as $T \uparrow \infty$. Thus, for example,

$$\sum_{k=1}^{\infty} |\langle \mathrm{q} G^{\alpha_1} \cdots \mathrm{q} G^{\alpha_m} e^{i\gamma S} \hat{v}_k, G^{*\alpha_0} e^{i\gamma S} \hat{v}_k \rangle|$$

$$\leqq \| \mathrm{q} G^{\alpha_1} \cdots \mathrm{q} G^{\alpha_{m-1}} \| \sum_{k=1}^{\infty} \| \mathrm{q} G^{\alpha_m} e^{i\gamma S} \hat{v}_k \| \| \mathrm{q} G^{*\alpha_0} e^{i\gamma S} \hat{v}_k \|$$

$$\leqq \| G \|^{\alpha_1 + \cdots + \alpha_{m-1}} \left\{ \sum_{k=1}^{\infty} \| \mathrm{q} G^{\alpha_m} e^{i\gamma S} \hat{v}_k \|^2 \sum_{j=1}^{\infty} \| \mathrm{q} G^{*\alpha_0} e^{i\gamma S} \hat{v}_j \|^2 \right\}^{1/2}$$

$$\leqq \| G \|^{\alpha_1 + \cdots + \alpha_{m-1}} 2\pi \left\{ \int_{-\infty}^{-2S} |x| |(g^{\alpha_m})^\vee(x)|^2 dx \int_{-\infty}^{-2S} |x| |(g^{*\alpha_0})^\vee(x)|^2 dx \right\}^{1/2}$$

$$= o(1),$$

as $T \uparrow \infty$; the bounds in the last two lines are taken from Corollary 3.1 and Lemma 3.7. A similar estimate shows that

$$\sum_{k=1}^{\infty} |\langle \mathrm{q} G^{\alpha_1} \cdots \mathrm{q} G^{\alpha_m} e^{i\gamma S} \hat{v}_k, G^{*\alpha_0} e^{i\gamma S} (\hat{v}_k + \hat{u}_k) \rangle| = o(1),$$

as $T \uparrow \infty$, to complete the proof.

Much the same sort of argument serves to prove

THEOREM 4.4. *If $\alpha_0, \alpha_1, \cdots, \alpha_m$ are positive integers which sum to $n + 1$, then*

$$U_T G^{\alpha_0} V_T G^{\alpha_1} V_T G^{\alpha_2} \cdots V_T G^{\alpha_m} U_T$$

is of trace class for $T \geqq R$ and the limit points of

$$\mathrm{trace}\{ U_T G^{\alpha_0} V_T G^{\alpha_1} \cdots V_T G^{\alpha_m} U_T \},$$

as $T \uparrow \infty$, are independent of h.

AMPLIFICATION. An elementary calculation shows that if $h^{\#}(\gamma) = h(-\gamma)$ and g is even and if ϕ_k, $k = 1,2,\cdots$, is an orthonormal basis for $M^T(d\Delta)$ which is chosen so that each term is either even or odd, then

$$\langle V_T G^{\alpha_1} V_T G^{\alpha_2} \cdots V_T G^{\alpha_m} \phi_k, G^{*\alpha_0} \phi_k \rangle_\Delta$$
$$= \langle W_T G^{\alpha_1} W_T G^{\alpha_2} \cdots W_T G^{\alpha_m} \phi_k, G^{*\alpha_0} \phi_k \rangle_\Delta.$$

(The fundamental fact here, as the referee has kindly pointed out, is that the mapping $f(\gamma) \to f(-\gamma)$ gives a unitary equivalence between $PG^{\alpha_0} VG^{\alpha_1} \cdots VG^{\alpha_m} P$ and $PG^{\alpha_0} WG^{\alpha_1} \cdots WG^{\alpha_m} P$.) Therefore the traces of the corresponding operators, which are independent of the choice of basis, must match:

$$\operatorname{trace} U_T G^{\alpha_0} V_T G^{\alpha_1} \cdots V_T G^{\alpha_m} U_T = \operatorname{trace} U_T G^{\alpha_0} W_T G^{\alpha_1} \cdots W_T G^{\alpha_m} U_T.$$

This gives an independent proof of Theorem 4.4 in the special case that $h^{\#}(\gamma) = h(-\gamma)$ and g is even.

The main theorem for UGU is now at hand:

THEOREM 4.5. *If g is a bounded uniformly continuous function of class $L^2(R^1, d\gamma)$, if $\tau + \tau' < \infty$ and if (1.3) is in effect, then*

$$(U_T G U_T)^n - U_T G^n U_T$$

is of trace class for every integer $n \geqq 1$ and all $T \geqq R$ and

$$\lim_{T \uparrow \infty} \operatorname{trace}[(U_T G U_T)^n - U_T G^n U_T] = -n \sum_{k=1}^{n-1} \int_0^\infty x \left(\frac{g^k}{k}\right)^{\vee}(x) \left(\frac{g^{n-k}}{n-k}\right)^{\vee}(-x)\,dx$$

independently of h.

PROOF. Theorem 4.2 insures that $(U_T G U_T)^n - U_T G^n U_T$ is of trace class for $n = 1, 2, \cdots$, and $T \geqq R$. The rest of the proof is perhaps best understood by focusing on the case $n = 3$. To begin with

$$\begin{aligned}(UGU)^3 - UG^3U &= UGUGUGU - UG^3U \\ &= -UG(V+W)G^2U - UG^2(V+W)GU \\ &\qquad + UG(V+W)G(V+W)GU,\end{aligned}$$

for $T \geqq R$. But as

$$\begin{aligned}|UGWGVGU|_1 &\leqq |UGW|_2 |WGVGU|_2 \\ &\leqq \|WGV\| |UGW|_2 |VGU|_2 = o(1)\end{aligned}$$

as $T \uparrow \infty$, in view of Lemmas 3.4 and 3.5, and (by the same argument)

$$|UGVGWGU|_1 = o(1)$$

as $T \uparrow \infty$, you see that

$$\begin{aligned}\operatorname{trace}\{(UGU)^3 - UG^3U\} &= \operatorname{trace}\{-UG(V+W)G^2U \\ &\qquad - UG^2(V+W)GU + UGVGVGU + UGWGWGU\} + o(1),\end{aligned}$$

as $T \uparrow \infty$. Theorems 4.3 and 4.4 now permit you to conclude that the limit points of

$$\operatorname{trace}\{(U_TGU_T)^3 - U_TG^3U_T\}$$

as $T \uparrow \infty$ are independent of h. They may therefore be evaluated by choosing $h = 1$. In that case $U_T = P_T$ for all $T \geqq 0$, $U_TG^3U_T$ is of trace class (as will follow from Lemma 5.1) and, by (1.6),

$$\begin{aligned}\operatorname{trace} U_TG^3U_T &= \operatorname{trace} P_TG^3P_T\\ &= 2T(g^3)^\vee(0) = \frac{T}{\pi}\int [g(\gamma)]^3 d\gamma.\end{aligned}$$

Moreover, by another application of (1.6),

$$\begin{aligned}&\operatorname{trace}\{(U_TGU_T)^3 - U_TG^3U_T\} = \operatorname{trace}\{(P_TGP_T)^3 - P_TG^3P_T\}\\ &= \iiint_{-T}^{T} g^\vee(x_1 - x_2)g^\vee(x_2 - x_3)g^\vee(x_3 - x_1)dx_1dx_2dx_3 - \frac{T}{\pi}\int [g(\gamma)]^3 d\gamma\end{aligned}$$

and, as $T \uparrow \infty$, this tends to the limit indicated in the statement of the theorem, as follows from the refined version of Kac's formula; see, e.g., theorem 5.7 of Devinatz [7]. This completes the proof for the case $n = 3$ since the existence of a limit is equivalent to there being only one limit point. Much the same sort of argument works for general n.

Amplification. To extract the trace formula in the special case $h = 1$ from theorem 5.7 of Devinatz you should identify the function k $[\hat{k}]$ in Devinatz with εg $[\varepsilon g^\vee]$ and check first that if ε is small enough and if say g and $g^\vee$ are summable and $\tau + \tau' < \infty$, then both the permanent assumption (4.5) of Devinatz and the hypotheses of his theorem 5.7 are met. The requisite formulas for such functions g then follow by matching powers of ε. Now if g is only subject to the hypotheses of our Theorem 4.5, then

$$g_\theta(\gamma) = \frac{\frac{1}{2\theta}\int_{-\theta}^{\theta} g(\gamma + \eta)d\eta}{1 + \theta^2\gamma^2} = \frac{\int e^{i\gamma x}g^\vee(x)\frac{\sin\theta x}{\theta x}dx}{1 + \theta^2\gamma^2},$$

in addition to meeting the same hypotheses, is also summable for every $\theta > 0$ as is

$$g_\theta^\vee(x) = \int g^\vee(x - y)(2\theta)^{-1}\exp(-|y|\theta^{-1})dy.$$

Therefore

$$\lim_{T\uparrow\infty} \operatorname{trace}\{(P_TG_\theta P_T)^n - P_TG_\theta^nP_T\} = \kappa_n(g_\theta)$$

in which $\kappa_n(g)$ designates the limit claimed in the statement of the Theorem and P_T is the projection onto $I^T(d\gamma)$. The same results are then obtained for g by letting $\theta \downarrow 0$. We shall sketch the proof for $n = 3$. To justify passing from G_θ to G on the left it is enough to show that

$$|PG(V+W)GPGP - PG_\theta(V+W)G_\theta PG_\theta P|_1 = o(1)$$

and

$$|PG^2(V+W)GP - PG_\theta^2(V+W)G_\theta P|_1 = o(1)$$

uniformly in T as $\theta \downarrow 0$, since

$$(PGP)^3 - PG^3P = -PG(V+W)GPGP - PG^2(V+W)GP.$$

But the second of these is equal to

$$\begin{aligned}&|P(G^2-G_\theta^2)(V+W)GP + PG_\theta^2(V+W)(G-G_\theta)P|_1\\ &\quad\leqq |P(G^2-G_\theta^2)(V+W)|_2|(V+W)GP|_2 + |PG_\theta^2(V+W)|_2|(V+W)\\ &\qquad\qquad \times(G-G_\theta)P|_2\end{aligned}$$

which tends to 0 uniformly in T as $\theta \downarrow 0$ in view of Lemma 3.4 and the fact that

(i) $\|g^k - g_\theta^k\|_\infty \to 0$

and

(ii) $\int |x||(g^k - g_\theta^k)^\vee(x)|^2 dx \to 0$

for $k = 1, 2, \cdots$, as $\theta \downarrow 0$. The first is disposed of in much the same way. The justification of (i) depends upon the fact that $\lim_{|\gamma|\uparrow\infty}|g(\gamma)| = 0$; see page 115 of Devinatz [7] for details on the latter. The justification of (ii) is tedious but straightforward (with the help of (i)). It now remains only to check that

$$\lim_{\theta\downarrow 0} \kappa_n(g_\theta) = \kappa_n(g)$$

but that is an immediate consequence of (ii).

5. Principal conclusions for *PGP*

The present objective is to study the trace of P_TGP_T. Recall that if $|h|^{-2}$ is locally summable, then, by Theorem 2.1, $I^T(d\Delta) = M^T(d\Delta)$ for every $T \geqq 0$.

Consequently the results derived in Section 4 for U_T become applicable to P_T also and can in fact be put into a more concrete form since the trace of P_TGP_T may be expressed explicitly in terms of $J_\gamma^T(\gamma)$:

LEMMA 5.1. *If* $T \geqq 0$, *and if*

$$\int |g(\gamma)| J_\gamma^T(\gamma) d\Delta(\gamma) < \infty,$$

then the operator $(P_TGP_T)^n$ *is of trace class for* $n = 1, 2, \cdots$, *and*

$$\text{trace } P_TGP_T = \frac{1}{\pi} \int g(\gamma) J_\gamma^T(\gamma) d\Delta(\gamma).$$

PROOF. Fix $T \geqq 0$, choose an orthonormal basis ϕ_k, $k = 1, 2, \cdots$, of $I^T(d\Delta)$ and let G° denote multiplication by $|g|$. Then, in view of identity (2.4),

$$\begin{aligned}\sum_{k=1}^{\infty} \langle P_TG^\circ P_T\phi_k, \phi_k\rangle_\Delta &= \sum_{k=1}^{\infty} \langle |g| \phi_k, \phi_k\rangle_\Delta \\ &= \int |g(\gamma)| J_\gamma^T(\gamma) d\Delta(\gamma).\end{aligned}$$

But now as $P_TG^\circ P_T$ is non-negative and the sum is finite by assumption it follows that $P_TG^\circ P_T$ is of trace class. Much the same sort of argument implies that $P_T(G^\circ - G_R)P_T$ and $P_T(G^\circ - G_I)P_T$ are of trace class, where $G_R[G_I]$ stands for multiplication by the real [imaginary] part of g. Hence

$$P_TGP_T = P_TG^\circ P_T - P_T[G^\circ - G_R]P_T + i\{P_TG^\circ P_T - P_T[G^\circ - G_I]P_T\}$$

and $(P_TGP_T)^n$, $n = 2, 3, \cdots$, are of trace class since that class is closed under addition and under multiplication by bounded operators. Moreover, another application of (2.4) yields the formula

$$\text{trace } P_TGP_T = \sum_{k=1}^{\infty} \langle G\phi_k, \phi_k\rangle_\Delta = \int g(\gamma) J_\gamma^T(\gamma) d\Delta(\gamma),$$

and the proof is complete.

AMPLIFICATION. If P_TGP_T is of trace class, then so is

$$P_LGP_L = P_L(P_TGP_T)P_L$$

for $0 \leqq L \leqq T$.

THEOREM 5.1. *If* (1.3) *is in effect and* $|h|^{-2}$ *is locally summable, if*

$$\int |g(\gamma)| J_\gamma^T(\gamma) d\Delta(\gamma) < \infty$$

for every $T \geqq 0$, if g is a bounded uniformly continuous function of class $L^2(R^1, d\gamma)$ and if

$$[\tau + \tau'] < \infty,$$

then $(P_T G P_T)^n$, $n = 1, 2, \cdots$, is of trace class for $T \geqq 0$ and

$$\lim_{T \uparrow \infty} \left\{ \operatorname{trace}(P_T G P_T)^n - \int [g(\gamma)]^n J_\gamma^T(\gamma) d\Delta(\gamma) \right\}$$

$$= -n \sum_{k=1}^{n-1} \int_0^\infty x \left(\frac{g^k}{k} \right)^\vee (x) \left(\frac{g^{n-k}}{n-k} \right)^\vee (-x) dx$$

independently of the choice of h, within the permissible class.

PROOF. Since $P_T = U_T$ for $T \geqq 0$, by Theorem 2.1, you have only to combine the implications of Theorem 4.5 and Lemma 5.1.

THEOREM 5.2. *If g enjoys the properties attributed to it in Theorem 5.1 and if $|\varepsilon|$ is sufficiently small, then*

$$\sum_{n=1}^\infty \frac{\varepsilon^n (P_T G P_T)^n}{n}$$

is of trace class for $T \geqq R$ and the determinant

$$\det(I - \varepsilon P_T G P_T) = \exp\left\{ -\operatorname{trace} \sum_{n=1}^\infty \frac{\varepsilon^n (P_T G P_T)^n}{n} \right\}$$

$$= \exp\left\{ \int \log[1 - \varepsilon g(\gamma)] J_\gamma^T(\gamma) d\Delta(\gamma) \right.$$

$$\left. + \int_0^\infty x (\log[1 - \varepsilon g])^\vee (x) (\log[1 - \varepsilon g])^\vee (-x) dx + o(1) \right\},$$

as $T \uparrow \infty$.

PROOF. Let

$$\alpha_n(T) = \operatorname{trace}\{(P_T G P_T)^n - P_T G^n P_T\}.$$

Then, by Theorem 4.2, you know that

$$|\alpha_n(T)| \leqq (n-1)^2 \|G\|^{n-2} [\tau + \tau']$$

for every $T \geqq R$. Therefore, by dominated convergence,

$$-\lim_{T \uparrow \infty} \sum_{n=1}^\infty \frac{\varepsilon^n \alpha_n(T)}{n} = -\sum_{n=1}^\infty \lim_{T \uparrow \infty} \frac{\varepsilon^n \alpha_n(T)}{n}$$

for $|\varepsilon| < (\|G\|)^{-1}$. But this is the same as to say that

$$\log\det(I - \varepsilon P_T G P_T) - \int \log[1 - \varepsilon g(\gamma)] J_\gamma^T(\gamma) d\Delta(\gamma)$$
$$= \int_0^\infty x(\log[1-\varepsilon g])^\vee(x)(\log[1-\varepsilon g])^\vee(-x)\,dx + o(1),$$

as $T \uparrow \infty$, as advertised.

It is worth emphasizing that under (1.2) both the conditions and conclusions of Theorems 5.1 and 5.2 can be expressed more concretely in terms of the phase ϑ of h:

COROLLARY 5.1. *If* (1.2) *is in effect, then you may set*

$$J_\gamma^T(\gamma) d\Delta(\gamma) = \frac{1}{\pi}[T + \vartheta'(\gamma)] d\gamma$$

for $T \geqq R$ *in both the hypotheses and conclusions of Theorems* 5.1 *and* 5.2. *Moreover,* $|h|^{-2}$ *is locally summable and* (1.3) *is fulfilled.*

PROOF. You have only to invoke Corollary 2.2 and Lemma 2.1.

AMPLIFICATION. The conclusions of Theorem 5.2 can be reformulated in a more classical vein since $P_T G P_T$ can be expressed as an integral operator K_T with kernel

$$K_T(\xi, \eta) = \int J_\xi^T(\gamma) g(\gamma) J_\gamma^T(\eta) d\Delta(\eta)$$

for real ξ and η, and the definition given for the determinant of $I - \varepsilon P_T G P_T$ coincides with the classical Fredholm determinant of $I - \varepsilon K_T$. It can also be expressed in terms of the eigenvalues $\lambda_j(T)$, $j = 1, 2, \cdots$ of $P_T G P_T$:

$$\det(I - \varepsilon P_T G P_T) = \prod_{j \geqq 1} [1 - \varepsilon \lambda_j(T)].$$

If $P_T G P_T$ has no eigenvalues, as is conceivably the case if $G \neq G^*$, then the product is just taken equal to 1. Gohberg–Krein [10] is suggested for additional information on such matters.

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